Simplest chaotic Hamiltonian

In [Sprott 2010, pp. 130 ff.] the simplest Hamiltonian exhibiting chaotic behavior is shown:

$$H = v^2 - u^2 - 2y2 - x^2 y \tag{1}$$

Applying the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

to equation (1) readily yields

$$\dot{x} = \frac{\partial H}{\partial v} = 2v,$$
$$\dot{v} = -\frac{\partial H}{\partial x} = 2xy,$$
$$\dot{y} = \frac{\partial H}{\partial u} = -2u, \text{ and}$$
$$\dot{u} = -\frac{\partial H}{\partial y} = 4y + x^2.$$

Combining those yields the following two coupled differential equations of second order:

$$\ddot{x} = 4xy$$
$$\ddot{y} = -8y - 2x^2$$

Suitable scaling factors for these equations are

$$\lambda_{\dot{x}} = 20,$$
$$\lambda_x = 5,$$
$$\lambda_{\dot{y}} = 5, \text{ and}$$
$$\lambda_y = 10,$$

yielding the final problem equations ready to be implemented on an analog computer such as THE ANALOG THING

$$\begin{aligned}\dot{x} &= \int 1.6xy\,\mathrm{d}t + 0.2,\\ x &= \int 0.25\dot{x}\,\mathrm{d}t + 0.2,\\ \dot{y} &= \int -4xy - 0.4x^2\,\mathrm{d}t + 0.55, \text{ and}\\ y &= \int 2y\,\mathrm{d}t - 0.4.\end{aligned}$$

The resulting program is shown in figure 1. Four typical phase space plots (with different sensitivity settings on the oscilloscope) are shown in figures 2, 3, 4, and 5.

Happy analog computing!

References

[Sprott 2010] Julien Clinton Sprott, *Elegant Chaos – Algebraically Simple Chaotic Flows*, World Scientific, 2010

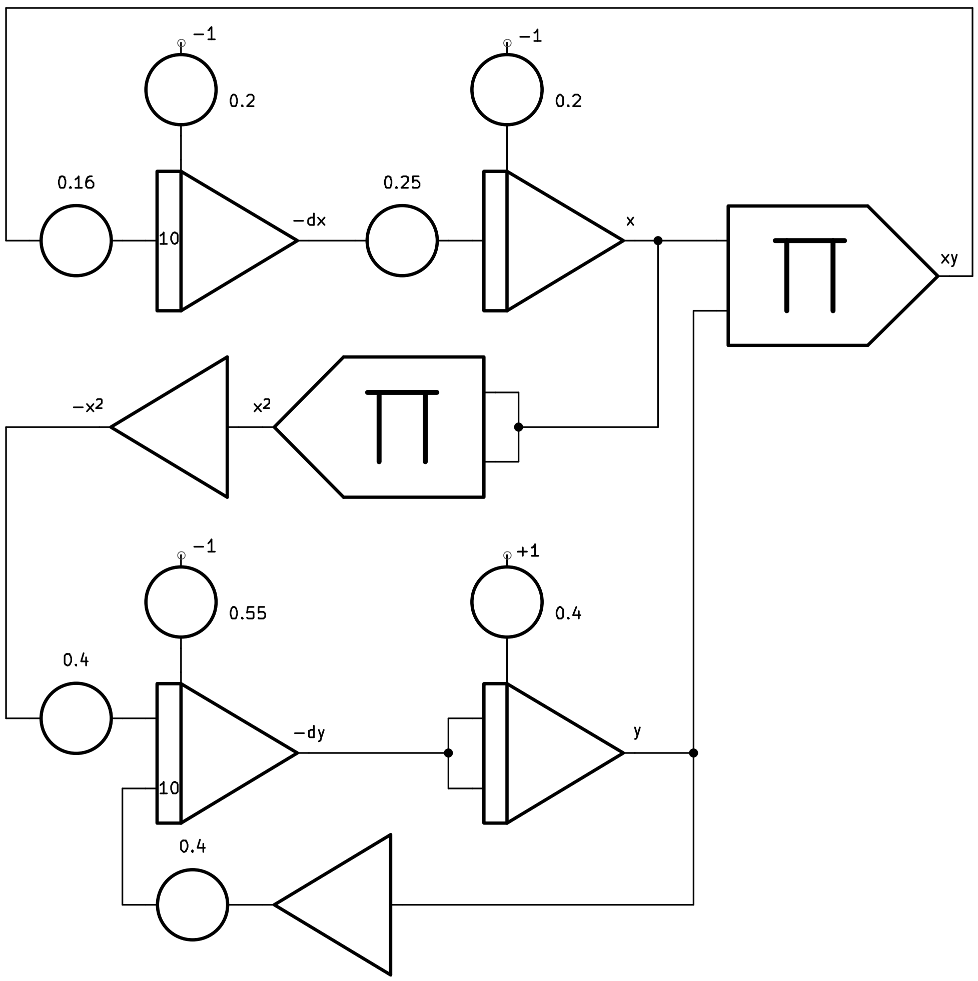

Figure 1: Program for the minimal chaotic Hamiltonian

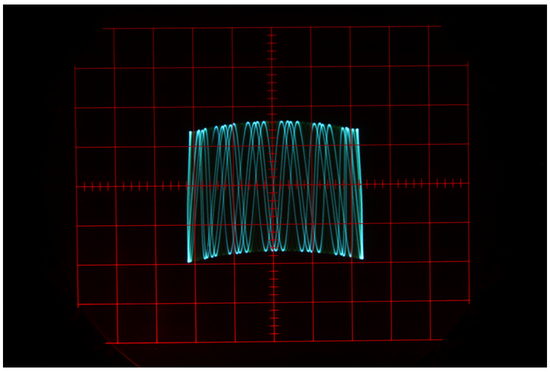

Figure 2: x/y phase space plot

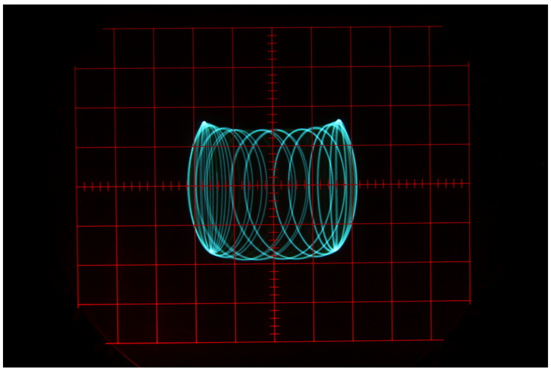

Figure 3: $-\dot{x}/y$ phase space plot

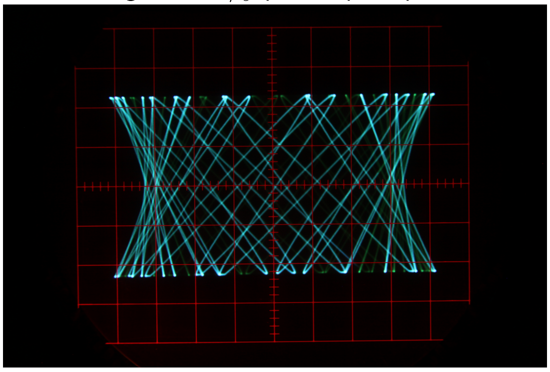

Figure 4: $-\dot{x}/-\dot{y}$ phase space plot

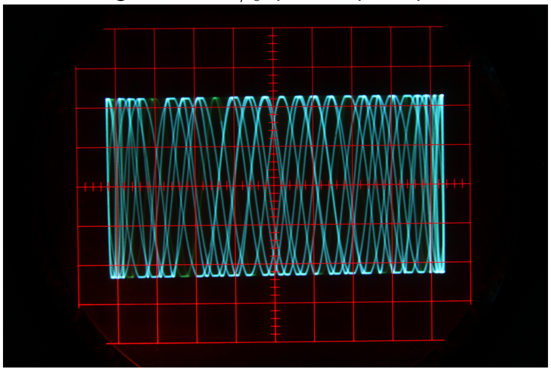

Figure 5: $x/-\dot{y}$ phase space plot