A chaotic system based on $|x|$

An interesting class of systems exhibiting chaotic behavior are *jerk systems*, autonomous differential equations of third order of the general form i

$$\dddot{x} = J(x, \dot{x}, \ddot{x}).$$

Here, $\dddot{x}$ is the *jerk*, i. e., the change in acceleration when the equation is assumed to describe a mechanical system, while $J(x, \dot{x}, \ddot{x})$ is the *jerk function* respectively. The simplest such system has been described in [Sprott 1997] and is of the form

$$\dddot{x} + A\ddot{x} - \dot{x}^2 + x = 0.$$

Based on this, [Linz et al. 1999] introduced a jerk system employing an absolute value function:[1]

$$\dddot{x} + A\ddot{x} + \dot{x} - |x| + 1 = 0. \tag{1}$$

With $A = 0.6$, all derivatives of x are well within $[-3, 3]$ and can be scaled with a common scaling factor $\lambda = \frac{1}{3}$ which cancels out in all terms but the constant 1 which gets scaled down to $\frac{1}{3}$. The corresponding analog computer program is shown in figure 1.[2] The necessary absolute value function can be implemented either with a comparator or with the circuit shown in figure 2.[3] The open amplifier on the left is used to implement an idealized diode while the amplifier on the right sums twice the output of this ideal diode to one times the input signal. The resulting phase space plots are shown in figure 3.

References

[Kiers et al. 2003] Ken Kiers, Tim Klein, Jeff Kolb, Steve Price, J. C. Sprott, "Chaos in a nonlinear analog computer", in *International Journal of Bifurcation and Chaos*, Vol. 14, No. 8, 2004, pp. 2867–2873

[1] Letting $\dot{x} = y$ and $\dot{y} = z$ this system can be written as three coupled differential equations with $\dot{z} = -Az - y + |x| - 1$ which might look more familiar to the analog computer programmer.

[2] [Kiers et al. 2003] describes an electronic implementation of this system.

[3] See [Ulmann 2023, pp. 82 f.].

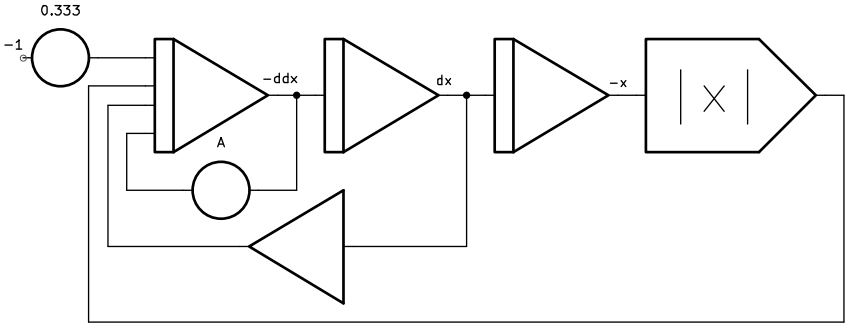

Figure 1: Implementation of equation (1)

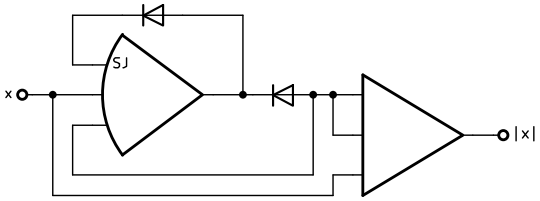

Figure 2: Implementation of $|x|$

[Linz et al. 1999] Stefan J. Linz, J. C. Sprott, "Elementary chaotic flow", in *Physics Letters A*, 256, 1999, pp. 240–245

[Sprott 1997] J. C. Sprott, "Simplest dissipative chaotic flow", in *Physics Letters A*, 228, 1997, pp. 271–273

[Ulmann 2023] Bernd Ulmann, *Analog and Hybrid Computer Programming*, 2nd edition, DeGruyter, 2023

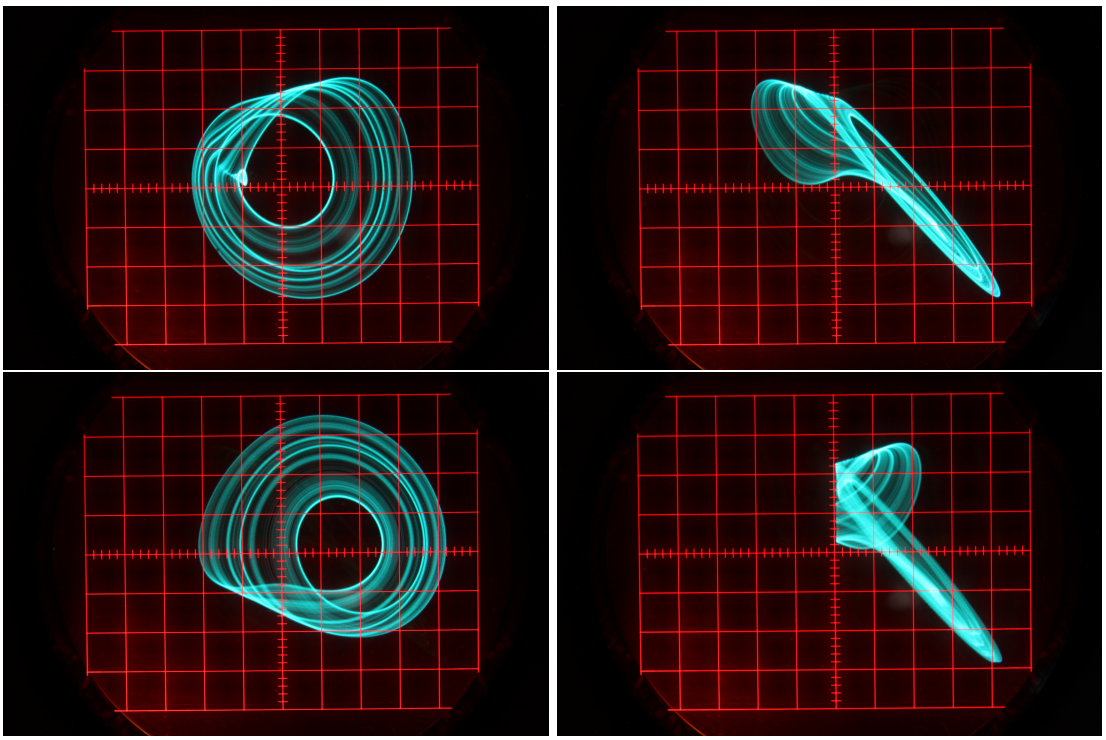

Figure 3: Phase space plots of the system (1) – from left to right and top to bottom the phase space plots show $-\ddot{x}/\dot{x}$, $-\ddot{x}/-m$, $\dot{x}/-x$, and $-\ddot{x}/|x|$