Three Time-Scale System

Multiple time scale dynamical systems are described by a system of coupled differential equations (DEQs) involving (vastly) different time scales. Typical examples are ample in areas such as biochemistry, chaotic systems, oscillators, neurophysiology, chemical reaction kinetics, etc.

Maybe the best overall introduction to this field is [KUEHN2015].[1] A simple such system involving two DEQs looks like

$$\begin{aligned} \varepsilon\dot{x} &= f(x, y, \varepsilon) \\ \dot{y} &= g(x, y, \varepsilon), \end{aligned}$$

where ε creates two different time scales. $\dot{x}$ and x are called *fast variables* while $\dot{y}$ and y are *slow variables*, accordingly. If ε is not too big, it can often be absorbed in a higher time scale factor k_0 of the integrators involved with the x variables.

Such multiple time scale systems often exhibit very interesting behavior such as spiking and bursting in the HINDMARSH-ROSE neuron model.[2] Particularly interesting (at least for the author) are *mixed-mode oscillations* (*MMO*s).[3] These are systems with periodic orbits exhibiting "peaks of substantially different amplitudes".[4] A well-known such system from chemistry is the BELOUSOV-ZHABOTINSKII reaction, a nonlinear chemical oscillator implementing a *chemical clock*.[5]

The following example contains three different time scales and is described by the coupled DEQs[6]

$$\begin{aligned} \dot{x} &= -y + c_2x^2 - c_3x^3, \\ \dot{y} &= \varepsilon(x - z) \text{ and} \\ \dot{z} &= \varepsilon^2(\mu - c_1y). \end{aligned}$$

[1] This is, in fact, one of my favorite books. . . :-)
[2] See https://analogparadigm.com/downloads/alpaca_28.pdf.
[3] See [KUEHN2015, pp. 398 ff.].
[4] See [KUEHN2015, p. 398].
[5] See [WINFREE 1984].
[6] See [KUEHN2015, p. 418].

ε and ε^2 implement two different time scales for the y and z variables, compared with the x variables. This system can be readily mechanized as shown in figure 1. It should be $\varepsilon \leq \frac{1}{10}$ for the system to exhibit an interesting behavior. μ is the primary bifurcation parameter and is in the range of several $\frac{1}{100}$. A good parameter set to start explorations from is

$$\varepsilon = \frac{1}{10},\ \mu = \frac{1}{25},\ c_1 = \frac{1}{2},\ c_2 = \frac{2}{5},\ \text{and } c_3 = \frac{1}{2}.$$

Figure 2 shows the behavior of this system for this particular set of parameters.[7] Playing with the parameters shows the richness of this system, which can exhibit various patterns of spikes as well as regular oscillations.

Happy analog computing! :-)

References

[Kuehn2015] Christian Kuehn, *Multiple Time Scale Dynamics*, Springer, 2015

[Winfree 1984] A. T. Winfree, "The Prehistory of the Belousov-Zhabotinsky Oscillator", in *Journal of Chemical Education*, Volume 61, Number 8, August 1984, pp. 661–663

[7] Here, x is plotted against time.

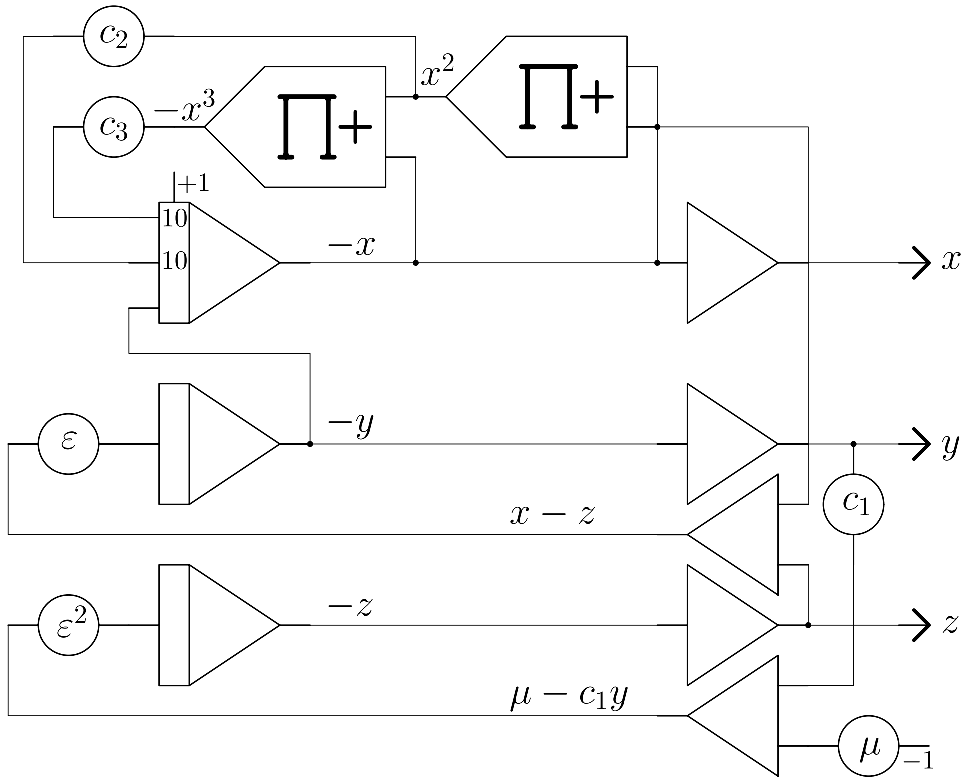

Figure 1: Analog computer setup for the three time scale system

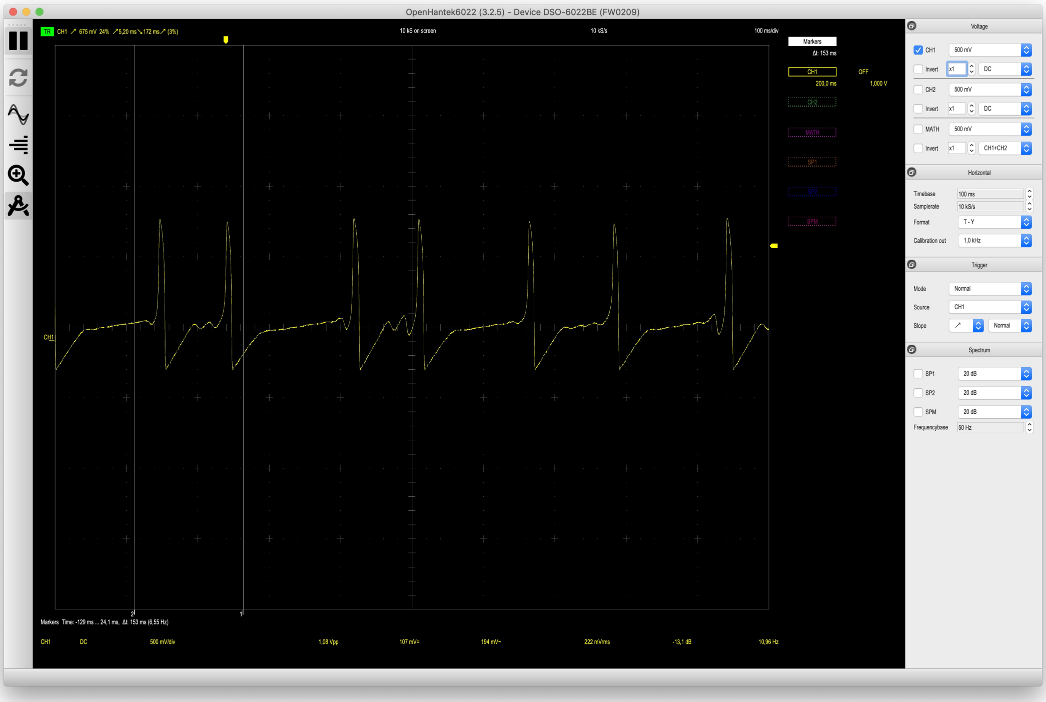

Figure 2: Behavior of the three time scale system