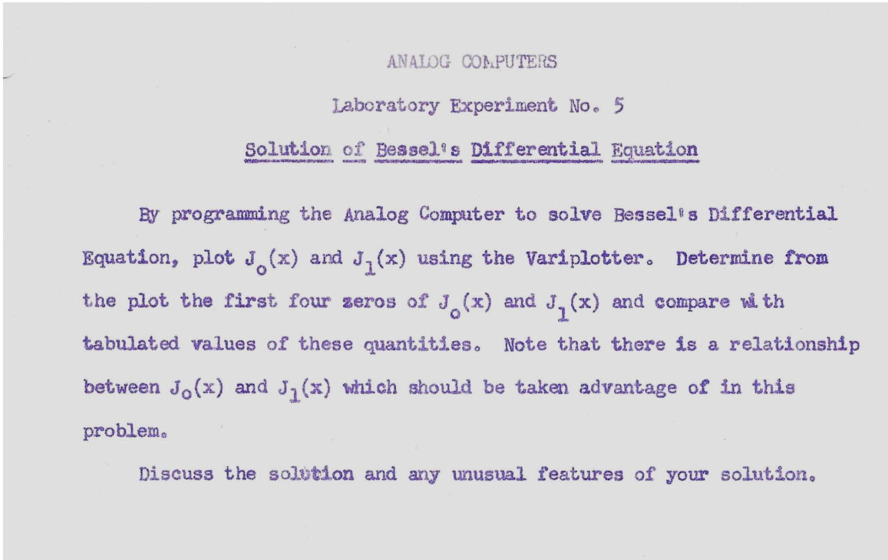
ANALOG COMPUTERS

Laboratory Experiment No. 5

Solution of Bessel's Differential Equation

By programming the Analog Computer to solve Bessel's Differential Equation, plot $J_o(x)$ and $J_1(x)$ using the Variplotter. Determine from the plot the first four zeros of $J_o(x)$ and $J_1(x)$ and compare with tabulated values of these quantities. Note that there is a relationship between $J_o(x)$ and $J_1(x)$ which should be taken advantage of in this problem.

Discuss the solution and any unusual features of your solution.

Figure 1: Classic Exercise

Generating Bessel functions

In June 2022, fellow analog computer enthusiast Dr. Chris Giles sent me a classic exercise shown in figure 1. This exercise is the basis of this application note.

Bessel *functions* were first described by Daniel Bernoulli[1] and later generalised by Friedrich Bessel.[2] Bessel functions of the first kind are usually denoted by $J_n(t)$ and are solutions of the Bessel differential equation

$$t^2\ddot{y} + t\dot{y} + (t^2 - n^2)y = 0. \tag{1}$$

Sometimes these are called *cylindrical harmonics*. The parameter n in the equation above defines the *order*. In the following, $n = 0$ and $n = 1$ are assumed.

[1] 01/27/1700–03/27/1782
[2] 07/22/1784–03/17/1846

For $n = 0$ equation (1) can be written as

$$\ddot{y} = -\frac{1}{t}\dot{y} - y$$

after dividing by t^2 and solving for $\ddot{y}$. This can be readily transformed into an analog computer program by applying KELVIN's feedback technique. The only thing to take into account is the term $\frac{1}{t}$ which is not well suited for an analog computer due to the pole at $t = 0$. It is far more easy to directly generate the quotient $\frac{\dot{y}}{t}$ as $\dot{y} \to 0$ with $t \to 0$.

The resulting program is shown in figure 2. Time t has been substituted by machine time τ which is generated using an integrator. The parameter $\dot{\tau}$ determines how fast τ rises and should be set so that $0 \leq \tau \leq 1$ during one computer run.

The relationship between $J_0(t)$ and $J_1(t)$ mentioned in the original exercise is an interesting one and can be found in [BRONSTEIN et al. 1989, p. 442] or any other standard textbook. In general

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-n}J_n(t)\right) = -t^{-n}J_{n+1}(t)$$

holds, which implies

$$J_1(t) = -\dot{J}_0(t)$$

for the case $n = 0$. Accordingly $J_1(\tau)$ is readily available in the program as it is just $-\dot{y}$.

Figure 3 shows the overall program setup on *THE ANALOG THING*.[3] A typical result is shown in figure 4. Note that the program has not been properly scaled. $\dot{\tau}$ was set according to the operate-time of the machine running in repetitive mode. The central scaling factor λ was set to get the desired result. ☺

[3] See `http://the-analog-thing.org`.

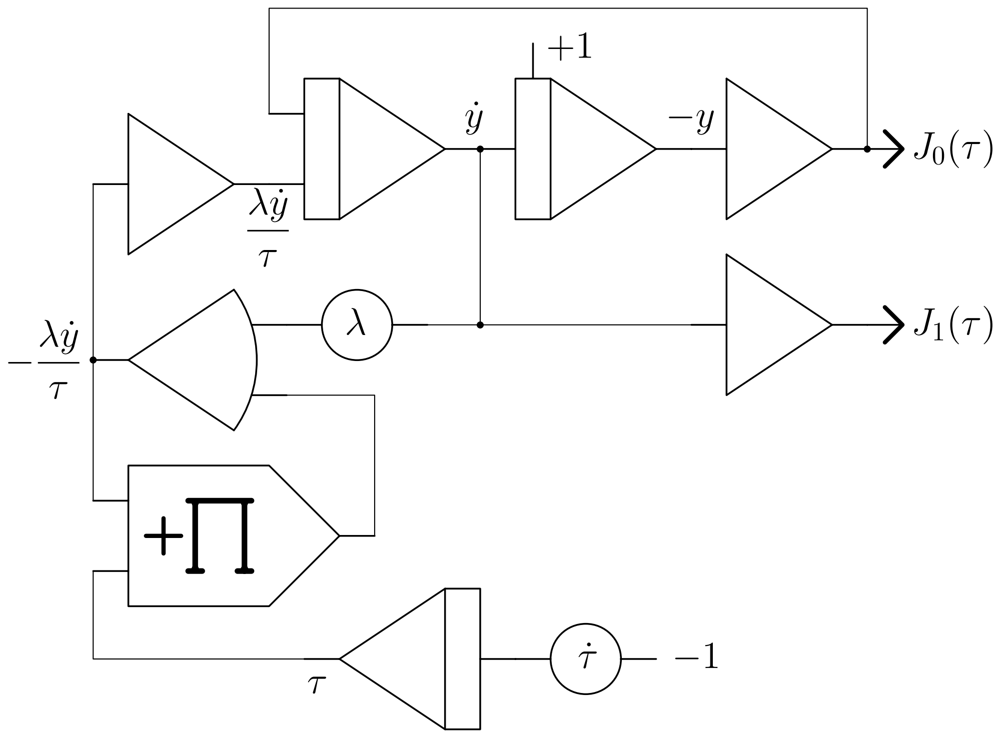

Figure 2: Analog computer program for generating Bessel functions $J_0(\tau)$ and $J_1(\tau)$

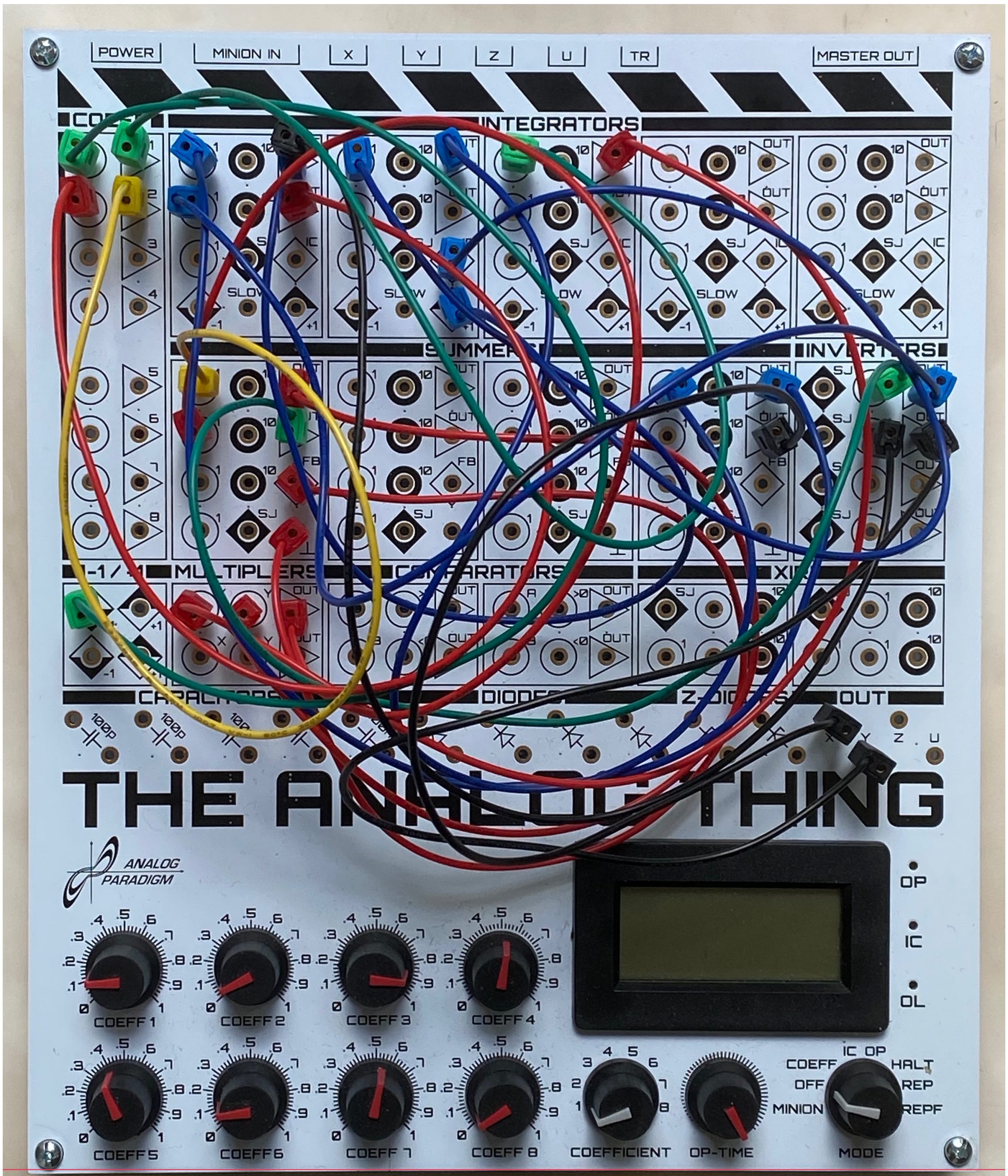

Figure 3: Setup for generating $J_0(\tau)$ and $J_1(\tau)$

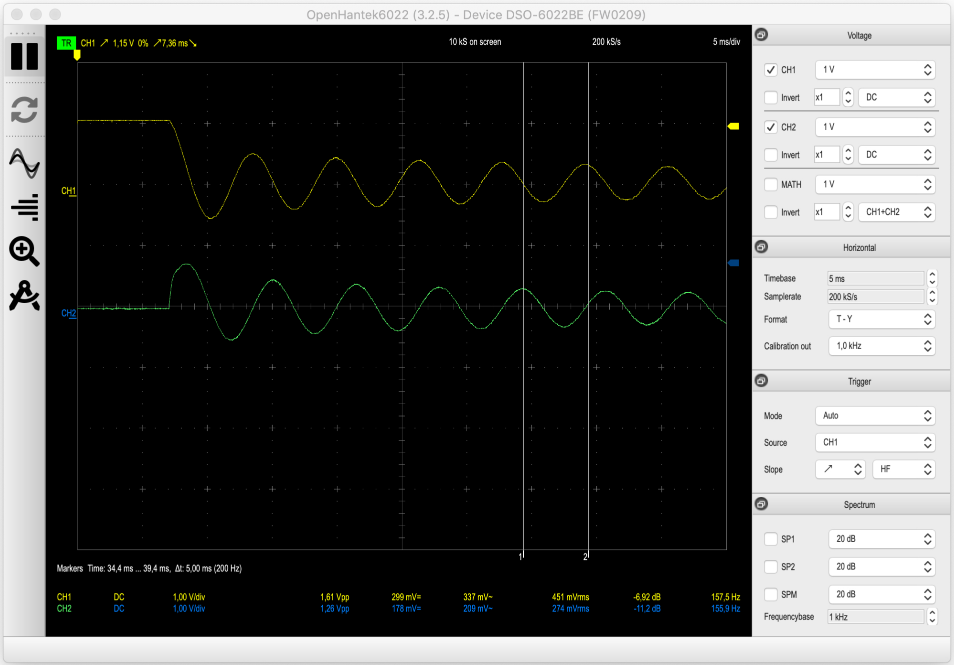

Figure 4: Typical output for $J_0(\tau)$ and $J_1(\tau)$

References

[BRONSTEIN et al. 1989] I. N. BRONSTEIN, K. A. SEMENDJAJEW, *Taschenbuch der Mathematik*, 24. Auflage, Verlag Harri Deutsch, Thun und Frankfurt/Main, 1989