The exponentially-mapped-past approach

It is often desirable to compute something like an arithmetic mean

$$\overline{x} = \frac{\sum\limits_{i=1}^{n} x_i}{n}$$

for a (time) continuous variable $x(t)$. A very simple approach could look like this:

$$\overline{x} = \frac{1}{t_1 - t_0} \int\limits_{t_0}^{t_1} x(t)\,\mathrm{d}t$$

Although this approach works fine on an analog computer it requires fixed times t_0 and t_1 which is only practical in few cases.

To overcome this problem [OTTERMAN(1960)] introduced *exponentially-mapped-past* (*EMP*) variables in order to extend the idea of a mean to continuous variables in continuous time on which the classic EAI application note [EAI 1.3.2(1964)] is based.[1] The basic idea is to introduce a weighting function that ensures that recent values influence the result more strongly than values in the past. The following equation demonstrates this technique with the integral running from the most distant past $-\infty$ to 0 (="now"):[2]

$$\widetilde{x}(0) = \alpha \int\limits_{-\infty}^{0} x(t)\mathrm{e}^{\alpha t}\,\mathrm{d}t$$

Here, α denotes a normalization factor:

$$\int\limits_{-\infty}^{0} \mathrm{e}^{\alpha t}\,\mathrm{d}t = \frac{1}{\alpha}$$

[1] OTTERMAN's work has its roots in [FANO(1950)]. This application note closely follows these sources as well as [EAI 1.3.2(1964)].

[2] $\widetilde{x}$ denotes the EMP mean.

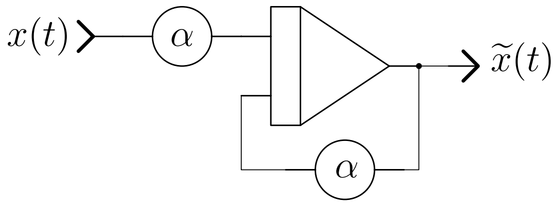

Figure 1: EMP mean circuit – the parameter α determines how quickly the weigthing function "forgets" past input values

Implementing this scheme on an analog computer is straightforward and guarantees that the integrator will not overload even with long run times (given that $x(t)$ remains in suitable bounds).

This can be further generalized as

$$\widetilde{x}(T) = \alpha \int\limits_{-\infty}^{T} x(t)\mathrm{e}^{\alpha(t-T)}\,\mathrm{d}t = \alpha\mathrm{e}^{-\alpha T}\int\limits_{-\infty}^{T} x(t)\mathrm{e}^{\alpha t}\,\mathrm{d}t, \tag{1}$$

a convolution integral of the input function $x(t)$ and an exponentially decaying weigthing function, the derivative of which with respect to T is

$$\frac{\mathrm{d}}{\mathrm{d}T}\widetilde{x} = \alpha\left(-\alpha\mathrm{e}^{-\alpha T}\int\limits_{-\infty}^{T} x(t)e^{\alpha t}\,\mathrm{d}t + \mathrm{e}^{-\alpha T}\mathrm{e}^{\alpha T}x(T)\right) = \alpha x(T) - \alpha\widetilde{x}(T). \tag{2}$$

Based on (2) the analog computer setup shown in figure 1 can be directly derived. This basically implements a "leaky integrator" which can also be seen as a low-pass RC filter. It should be noted that this only works if no exact estimation of the mean value is required during the startup time of the computation. After a step input, the output will reach 95% of the step height in the time interval $3/\alpha$. This must be taken into account for the startup time as well.

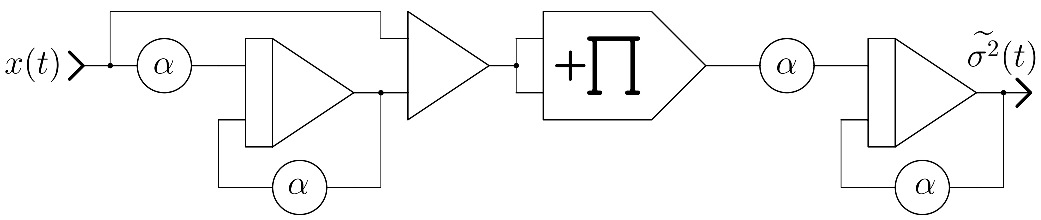

Figure 2: EMP variance circuit

One can now ask if the idea of EMP variables also allows the calculation of a type of variance. In the discrete case the variance is defined by

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \widetilde{x}\right)^2.$$

Analogously to (1) this can be extended to continuous variables in continuous time as

$$\widetilde{\sigma^2}(T) = \alpha \int\limits_{-\infty}^{T} \left(x(t) - \widetilde{x}(t)\right)^2 \mathrm{e}^{\alpha(t-T)}\,\mathrm{d}t = \alpha \mathrm{e}^{-\alpha T} \int\limits_{-\infty}^{T} \left(x(t) - \widetilde{x}(t)\right)^2 \mathrm{e}^{\alpha t}\,\mathrm{d}t$$

which can be mechanized by the analog computer setup shown in figure 2.

Equally straightforward is the computation of an EMP autocorrelation $\widetilde{\rho}(\tau)$ based on

$$\widetilde{\rho}(\tau) = \alpha \int\limits_{-\infty}^{T} x(t)x(t-\tau)\mathrm{e}^{-\alpha(T-t)}\,\mathrm{d}t$$

as shown in figure 3 where τ represents the time delay used for the correlation. The time delay function shown can be implemented using various techniques such as PADÉ or STUBBS-SINGLE approximations (cf. [ULMANN(2020), pp. 95–108]).

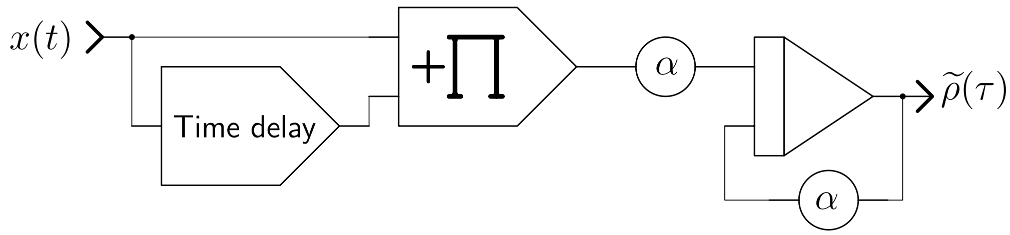

Figure 3: EMP autocorrelation circuit

The WIENER-KHINCHIN theorem states that the spectral decomposition of the autocorrelation function of a suitable function is given by the power spectrum of that function.[3] Thus, it is possible to compute an EMP power spectrum based on $\widetilde{\rho}(\tau)$.

The EMP FOURIER transform is defined as

$$\widetilde{F}(\omega)=\alpha\int\limits_{-\infty}^{T}x(t)\mathrm{e}^{-\alpha(T-t)}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t=\alpha\mathrm{e}^{-\mathrm{i}\omega t}\int\limits_{-\infty}^{T}x(t)\mathrm{e}^{-\alpha(T-t)}\mathrm{e}^{\mathrm{i}\omega(T-t)}\,\mathrm{d}t.$$

The power spectrum is $P(\omega)=|F(\omega)|^2$, i. e. in the EMP case it is

$$\widetilde{P}(\omega)=\alpha^2\left[\left(\int\limits_{-\infty}^{T}x(t)\mathrm{e}^{-\alpha(T-t)}\cos(\omega(T-t))\,\mathrm{d}t\right)^2+\left(\int\limits_{-\infty}^{T}x(t)\mathrm{e}^{-\alpha(T-t)}\sin(\omega(T-t))\,\mathrm{d}t\right)^2\right].$$

The corresponding analog computer setup is shown in figure 4. At its heart is a simple quadrature oscillator consisting of two integrators and a summer in a loop. This yields both, the sine and cosine components, the squares of which are summed to yield the desired output.

[3]Questions regarding convergence criteria are beyond the scope of this application note.

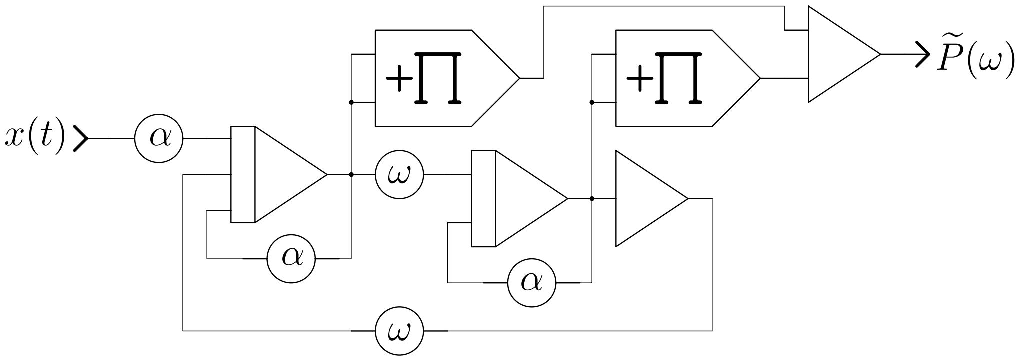

Figure 4: EMP FOURIER circuit

References

[EAI 1.3.2(1964)] EAI, *Continuous Data Analysis with Analog Computers Using Statistical and Regression Techniques*, EAI Applications Reference Library 1.3.2a, 1964, http://bitsavers.org/pdf/eai/applicationsLibrary/1.3.2a_Continuous_Data_Analysis_with_Analog_Computers_Using_Statistical_and_Regression_Techniques_1964.pdf

[FANO(1950)] R. M. FANO, "Short-Time Autocorrelation Functions and Power Spectra", in *The Journal of the Acoustical Society of America*, Volume 22, Number 5, pp. 546–550

[OTTERMAN(1960)] JOSEPH OTTERMAN, "The Properties and Methods for Computation of Exponentially-Mapped-Past Statistical Variables", in *IRE Transactions on Automatic Control*, Volume: AC-5, Issue: 1, 1. Jan. 1960, pp. 11–17, DOI: 10.1109/TAC.1960.6429289

[ULMANN(2020)] BERND ULMANN, *Analog and Hybrid Computer Programming*, DeGruyter, 2020