

## The Duffing Oscillator

A *Duffing equation* describes any oscillator featuring a cubic stiffness term, i. e. a nonlinear elasticity, such as

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = 0.$$

It is named after *Georg Wilhelm Christian Caspar Duffing* (1861 – 1944), a German engineer and inventor who wrote a seminal work on forced oscillations in 1918 (cf. [DUFFING 1918]).

This oscillator exhibits a nice chaotic behaviour when driven by an external forcing function i. e. in the form of

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega_0 t).$$

Thorough theoretical treatments of this oscillator may be found in [KORSCH et al. 2008], [HÖHLER 1988], and [CHANG 2017]. Apart from its interesting mathematical properties it is a nice dynamic system which can be implemented on an analog computer without complicated scaling and which invites to playing with parameters to achieve all kinds of nice phase space plots.

Figure 1 shows the implementation of the forcing function  $f(t) = \gamma \cos(\omega_0 t)$ . To achieve a minimum of harmonic distortion, the two amplitude limiting Zener diodes are connected to an integrator input with weight 1.  $\varepsilon$  introduces a tiny positive feedback signal to avoid a decreasing amplitude, while  $\gamma$  controls the amplitude of the output signal.

Figure 2 shows the straight-forward implementation of the DUFFING oscillator.  $\beta$  was scaled down by a factor of  $\frac{1}{10}$  to allow for values of  $\beta$  up to 10.

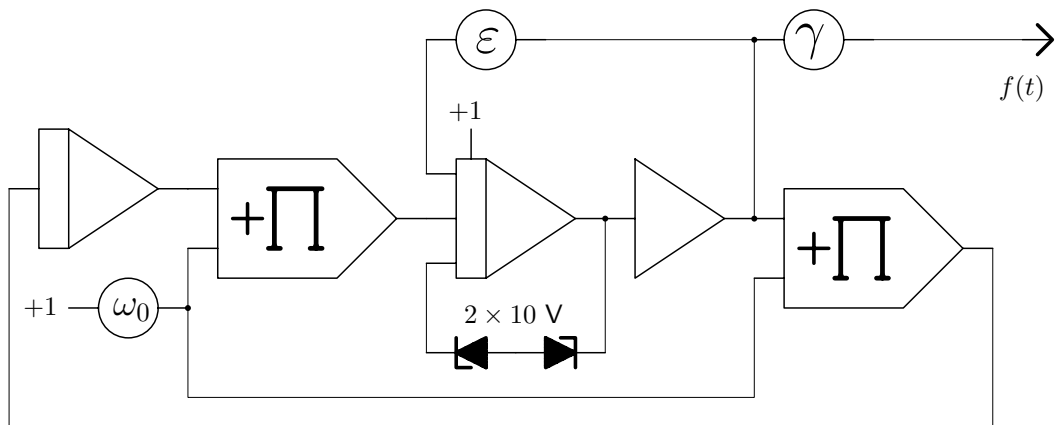


Figure 1: Forcing function  $f(t)$

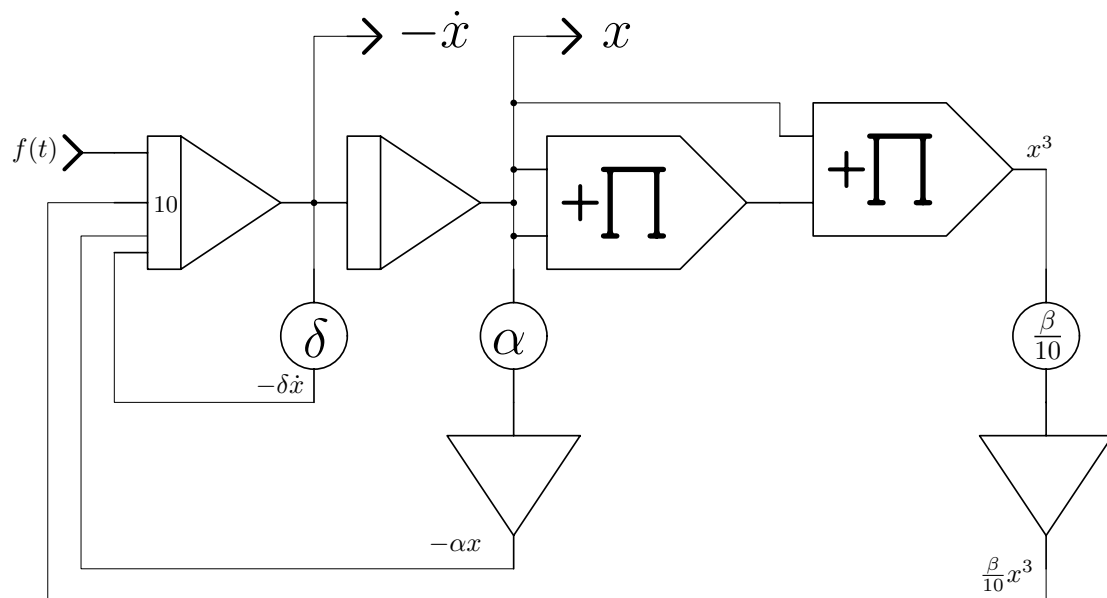


Figure 2: DUFFING oscillator



# Analog Computer Applications

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A good parameter set to start from is shown in the following:

$$\alpha = 1$$

$$\beta = 5$$

$$\gamma = 1$$

$$\delta = 0.02$$

$$\omega_0 = 1$$

$$\varepsilon = \text{as small as possible}$$

Varying the parameters manually shows the very rich behaviour of this forced oscillator which even shows chaotic characteristics. The effect of  $\omega_0$  is quite distinct. Two typical phase space plots generated by plotting  $x$  against  $-\dot{x}$  are shown in figure 3. To get a wider range for  $\gamma$  the output  $f(t)$  may also be connected to an input with weight 10 on the DUFFING oscillator sub-circuit.

## References

- [CHANG 2017] TAI-PING CHANG, "Chaotic Motion in Forced Duffing System Subject to Linear and Nonlinear Damping", in *Hindawi – Mathematical Problems in Engineering*, Volume 2017, Article ID 3769870, 15
- [DUFFING 1918] GEORG DUFFING, "Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung", Sammlung Vieweg, Heft 41/42, Braunschweig 1918
- [HÖHLER 1988] G. HÖHLER, "Schwingungen II", in *Computer – Theoretikum und Praktikum für Physiker*, Fachinformationszentrum Karlsruhe, 1988, pp. 5–73
- [KORSCH et al. 2008] HANS JÜRGEN KORSCH, HANS-JÖRG JODL, TIMO HARTMANN, "The Duffing Oscillator", in *Chaos – A Program Collection for the PC*, Springer, 2008, pp. 157–184

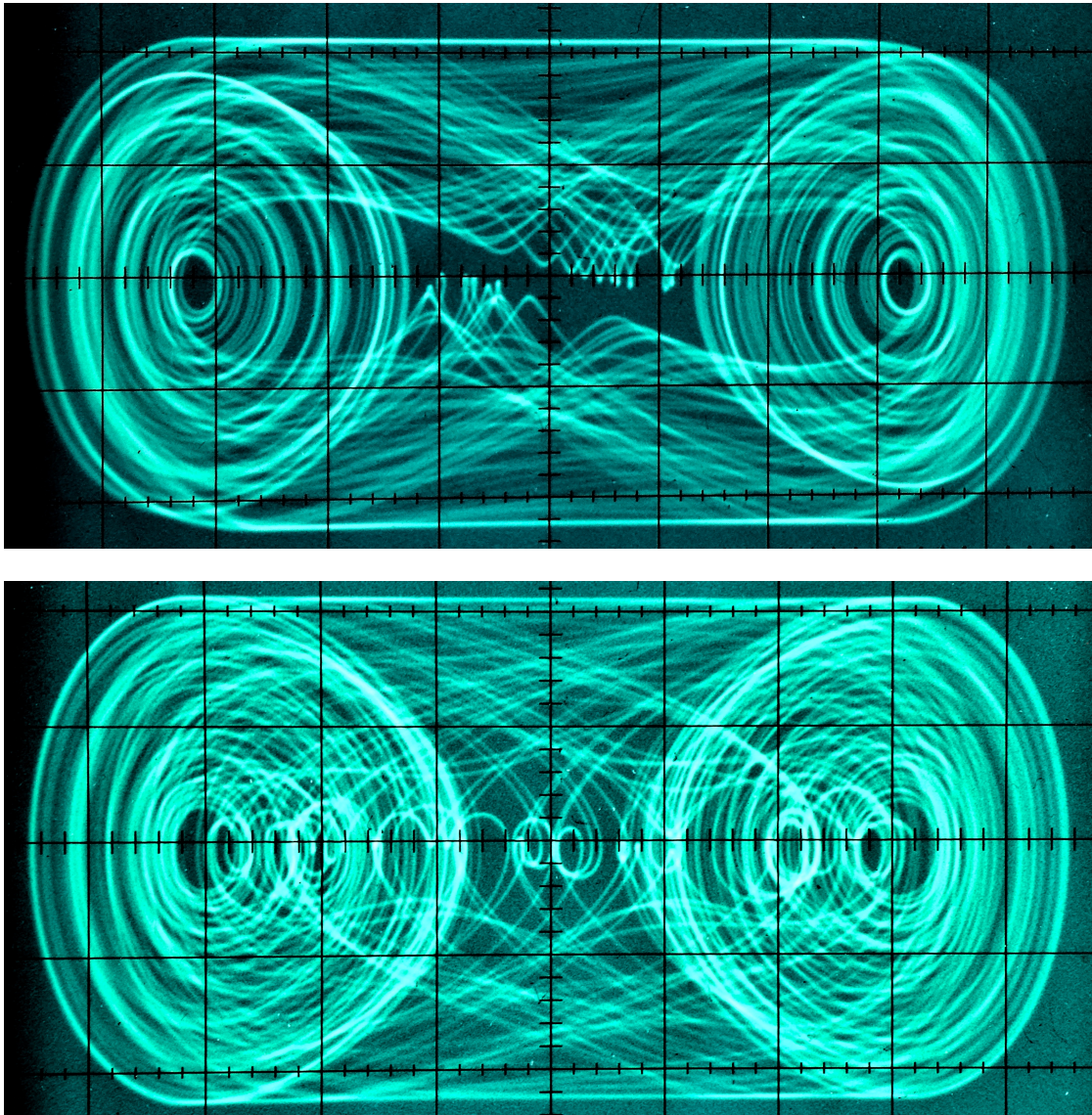


Figure 3: Two phase space plots showing the chaotic behaviour of the *DUFFING* oscillator